

FROM HOPF ALGEBRAS TO TENSOR CATEGORIES

N. ANDRUSKIEWITSCH, I. ANGIONO, A. GARCÍA IGLESIAS,
B. TORRECILLAS, C. VAY

ABSTRACT. This is a survey on spherical Hopf algebras. We give criteria to decide when a Hopf algebra is spherical and collect examples. We discuss tilting modules as a mean to obtain a fusion subcategory of the non-degenerate quotient of the category of representations of a suitable Hopf algebra.

INTRODUCTION

It follows from its very definition that the category $\text{Rep } H$ of finite-dimensional representations of a Hopf algebra H is a tensor category. There is a less obvious way to go from Hopf algebras with some extra structure (called spherical Hopf algebras) to tensor categories. Spherical Hopf algebras and the procedure to obtain a tensor category from them were introduced by Barrett and Westbury [BaW1, BaW2], inspired by previous work by Reshetikhin and Turaev [RT1, RT2], in turn motivated to give a mathematical foundation to the work of Witten [W]. A spherical Hopf algebra has by definition a group-like element that implements the square of the antipode (called a pivot) and satisfies the left-right trace symmetry condition (2.2). The classification (or even the characterization) of spherical Hopf algebras is far from being understood, but there are two classes to start with. Let us first observe that semisimple spherical Hopf algebras are excluded from our considerations, since the tensor categories arising from the procedure are identical to the categories of representations. Another remark: any Hopf algebra is embedded in a pivotal one, so that the trace condition (2.2) is really the crucial point. Now the two classes we mean are

- Hopf algebras with involutory pivot, or what is more or less the same, with $\mathcal{S}^4 = \text{id}$. Here the trace condition follows for free, and the quantum dimensions will be (positive and negative) integers.
- Ribbon Hopf algebras [RT1, RT2].

It is easy to characterize pointed or copointed Hopf algebras with $\mathcal{S}^4 = \text{id}$; so we have many examples of (pointed or copointed) spherical Hopf algebras

2000 *Mathematics Subject Classification.* 16W30.

The work of N. A., I. A., A. G. I. and C. V. was partially supported by CONICET, FONCyT-ANPCyT and Secyt (UNC). The work of B. T. was partially supported by MTM 2011-27090, Ministerio de Educación y Ciencia (España) and the project FQM 3128 from Junta Andalucía (FEDER).

with involutory pivot, most of them not even quasi-triangular, see Section 2.6. On the other hand, any quasitriangular Hopf algebra is embedded in a ribbon one [RT1]; combined with the construction of the Drinfeld double, we see that any finite-dimensional Hopf algebra gives rise to a ribbon one. So, we have plenty of examples of spherical Hopf algebras.

The procedure to get a tensor category from a spherical Hopf algebra H consists in taking a suitable quotient $\underline{\text{Rep}} H$ of the category $\text{Rep } H$. This appears in [BaW1] but similar ideas can be found elsewhere, see e. g. [GK, Kn]. The resulting spherical categories are semisimple but seldom have a finite number of irreducibles, that is, they are seldom fusion categories in the sense of [ENO]. We are interested in describing fusion tensor subcategories of $\underline{\text{Rep}} H$ for suitable H . This turns out to be a tricky problem. First, if the pivot is involutive, then the fusion subcategories of $\underline{\text{Rep}} H$ are integral, see Proposition 2.12. The only way we know is through tilting modules; but it seems to us that there is no general method, just a clever recipe that works. This procedure has a significant outcome in the case of quantum groups at roots of one, where the celebrated Verlinde categories are obtained [AP]; see also [Sa] for a self-contained exposition and [M] for similar results in the setting of algebraic groups over fields of positive characteristic. One should also mention that the Verlinde categories can be also constructed from vertex operator algebras related to affine Kac-moody algebras, see [BaK, Hu, HuL, KaLu] and references therein; the comparison of these two approaches is highly non-trivial. Another approach, at least for $SL(n)$, was proposed in [Hay] via face algebras (a notion predecessor of weak Hopf algebras).

The paper is organized as follows. Section 1 contains some information about the structure of Hopf algebras and notation used later in the paper. Section 2 is devoted to spherical Hopf algebras. In Section 3 we discuss tilting modules and how this recipe would work for some finite-dimensional pointed Hopf algebras associated to Nichols algebras of diagonal type, that might be thought of as generalizations of the small quantum groups of Lusztig.

1. PRELIMINARIES

1.1. Notations. Let \mathbb{k} be an algebraically closed field of characteristic 0 and \mathbb{k}^\times its multiplicative group of units. All vector spaces, algebras, unadorned Hom and \otimes are over \mathbb{k} . By convention, $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Let G be a finite group. We denote by $Z(G)$ the center of G and by $\text{Irr } G$ the set of isomorphism classes of irreducible representations of G . If $g \in G$, we denote by $C_G(g)$ the centralizer of g in G . The conjugacy class of g is denoted by \mathcal{O}_g or by \mathcal{O}_g^G , when emphasis on the group is needed.

Let A be an algebra. We denote by $Z(A)$ the center of A . The category of A -modules is denoted $A\text{-Mod}$; the full subcategory of finite-dimensional objects is denoted $A\text{-mod}$. The set of isomorphism classes of irreducible¹

¹that is, minimal, see page 8.

objects in an abelian category \mathcal{C} is denoted $\text{Irr } \mathcal{C}$; we use the abbreviation $\text{Irr } A$ instead of $\text{Irr } A\text{-mod}$.

1.2. Tensor categories. We refer to [BaK, McL, EGNO, ENO, EO] for basic results and terminology on tensor and monoidal categories. A monoidal category is one with tensor product and unit, denoted $\mathbf{1}$; thus $\text{End}(\mathbf{1})$ is a monoid. A monoidal category is rigid when it has right and left dualities. In this article, we understand by tensor category a monoidal rigid abelian \mathbb{k} -linear category, with $\text{End}(\mathbf{1}) \simeq \mathbb{k}$. A particular important class of tensor categories is that of fusion categories, that is semisimple tensor categories with finite set of isomorphism classes of simple objects, that includes the unit object, and finite-dimensional spaces of morphisms. Another important class of tensor categories is that of braided tensor categories, i. e. those with a commutativity constraint $c_{V,W} : V \otimes W \rightarrow W \otimes V$ for every objects V and W . A braided vector space is a pair (V, c) where V is a vector space and $c \in GL(V \otimes V)$ satisfies $(c \otimes \text{id})(\text{id} \otimes c)(c \otimes \text{id}) = (\text{id} \otimes c)(c \otimes \text{id})(\text{id} \otimes c)$; this notion is closely related to that of braided tensor category.

1.3. Hopf algebras. We use standard notation for Hopf algebras (always assumed with bijective antipode); $\Delta, \varepsilon, \mathcal{S}$, denote respectively the comultiplication, the counit, and the antipode. For the first, we use the Heyneman-Sweedler notation $\Delta(x) = x_{(1)} \otimes x_{(2)}$. The tensor category of finite-dimensional representations of a Hopf algebra H is denoted $\text{Rep } H$ instead of $H\text{-mod}$, to stress the tensor structure. There are two duality endo-functors of $\text{Rep } H$ composing the transpose of the action with either the antipode or its inverse: if $M \in \text{Rep } H$, then $M^* = \text{Hom}(M, \mathbb{k}) = {}^*M$, with actions

$$\langle h \cdot f, m \rangle = \langle f, \mathcal{S}(h) \cdot m \rangle, \quad \langle h \cdot g, m \rangle = \langle g, \mathcal{S}^{-1}(h) \cdot m \rangle,$$

for $h \in H, f \in M^*, g \in {}^*M, m \in M$.

The tensor category of finite-dimensional corepresentations of a Hopf algebra H , i. e. right comodules, is denoted $\text{Corep } H$. The coaction map of $V \in \text{Corep } H$ is denoted $\rho = \rho_V : V \rightarrow V \otimes H$; in Heyneman-Sweedler notation, $\rho(v) = v_{(0)} \otimes v_{(1)}, v \in V$. Also, the coaction map of a left comodule W is denoted $\delta = \delta_W : W \rightarrow W \otimes H$; that is, $\delta(w) = w_{(-1)} \otimes w_{(0)}, w \in W$.

A Yetter-Drinfeld module V over a Hopf algebra H is simultaneously a left H -module and a left H -comodule, subject to the compatibility condition $\delta(h \cdot v) = h_{(1)} v_{(-1)} \mathcal{S}(h_{(3)}) \otimes h_{(2)} \cdot v_{(0)}$ for $v \in V, h \in H$. The category ${}^H_H \mathcal{YD}$ of Yetter-Drinfeld modules over H is a braided tensor category with braiding $c(v \otimes w) = v_{(-1)} \cdot w \otimes v_{(0)}$, see e. g. [AS2, Mo]; when $\dim H < \infty$, ${}^H_H \mathcal{YD}$ coincides with the category of representations of the Drinfeld double $D(H)$.

Let H be a Hopf algebra. A basic list of H -invariants is

- The group $G(H)$ of group-like elements of H ,
- the coradical $H_0 =$ largest cosemisimple subcoalgebra of H ,
- the coradical filtration of H .

Assume that H_0 is a Hopf subalgebra of H . In this case, another fundamental invariant of H is the *infinitesimal braiding*, a Yetter-Drinfeld module V over H_0 , see [AS2]. We shall consider two particular cases:

- The Hopf algebra H is *pointed* if $H_0 = \mathbb{k}G(H)$.
- The Hopf algebra H is *copointed* if $H_0 = \mathbb{k}^G$ for a finite group G .

Recall that $x \in H$ is $(g, 1)$ skew-primitive when $\Delta(x) = x \otimes 1 + g \otimes x$; necessarily, $g \in G(H)$. If H is generated as an algebra by group-like and skew-primitive elements, then it is pointed.

We shall need the description of *Yetter-Drinfeld modules* over $\mathbb{k}G$, G a finite group; these are G -graded vector spaces $M = \bigoplus_{g \in G} M_g$ provided with a G -module structure such that $g \cdot M_t = M_{gtg^{-1}}$ for any $g, t \in G$. The category ${}_{\mathbb{k}G}^G \mathcal{YD}$ of Yetter-Drinfeld modules over G is semisimple and its irreducible objects are parameterized by pairs (\mathcal{O}, ρ) , where \mathcal{O} is a conjugacy class of G and $\rho \in \text{Irr } C_G(g)$, $g \in \mathcal{O}$ fixed. We describe the corresponding irreducible Yetter-Drinfeld module $M(\mathcal{O}, \rho)$. Let $g_1 = g, \dots, g_m$ be a numeration of \mathcal{O} and let $x_i \in G$ such that $x_i g x_i^{-1} = g_i$ for all $1 \leq i \leq m$. Then

$$M(\mathcal{O}, \rho) = \text{Ind}_{C_G(g)}^G V = \bigoplus_{1 \leq i \leq m} x_i \otimes V.$$

Let $x_i v := x_i \otimes v \in M(\mathcal{O}, \rho)$, $1 \leq i \leq m$, $v \in V$. The Yetter-Drinfeld module $M(\mathcal{O}, \rho)$ is a braided vector space with braiding given by

$$(1.1) \quad c(x_i v \otimes x_j w) = g_i \cdot (x_j w) \otimes x_i v = x_h \rho(\gamma)(w) \otimes x_i v$$

for any $1 \leq i, j \leq m$, $v, w \in V$, where $g_i x_j = x_h \gamma$ for unique h , $1 \leq h \leq m$, and $\gamma \in C_G(g)$. Now the categories ${}_{\mathbb{k}G}^G \mathcal{YD}$ and ${}_{\mathbb{k}G}^G \mathcal{YD}$ are tensor equivalent, so that a similar description of the objects of the latter holds, see e. g. [AV1].

The following notion is appropriate to describe all braided vector spaces arising as Yetter-Drinfeld modules over some finite abelian group. A braided vector space (V, c) is of *diagonal type* if there exist $q_{ij} \in \mathbb{k}^\times$ and a basis $\{x_i\}_{i \in I}$ of V such that $c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i$, for each pair $i, j \in I$. In such case, we say that $i, j \in I$ are *connected* if there exist $i_k \in I$, $k = 0, 1, \dots, n$, such that $i_0 = i$, $i_n = j$, and $q_{i_{k-1}, i_k} q_{i_k, i_{k+1}} \neq 1$, $1 \leq k \leq n$. It establishes an equivalence relation on I . The equivalence classes are called connected components, and V is *connected* if it has a unique component.

When $H = \mathbb{k}\Gamma$, where Γ is a finite abelian group, each $V \in {}_H^H \mathcal{YD}$ is a braided vector space of diagonal type. Indeed, $V = \bigoplus_{g \in \Gamma, \chi \in \widehat{\Gamma}} V_g^\chi$, where $V_g = \{v \in V \mid \delta(v) = g \otimes v\}$, $V^\chi = \{v \in V \mid g \cdot v = \chi(g)v \text{ for all } g \in \Gamma\}$, $V_g^\chi = V^\chi \cap V_g$. Note that $c(x \otimes y) = \chi(g)y \otimes x$, for each $x \in V_g$, $g \in \Gamma$, $y \in V^\chi$, $\chi \in \widehat{\Gamma}$. On the other hand, we can realize every braided vector space of diagonal type as a Yetter-Drinfeld module over the group algebra of an appropriate abelian group.

1.4. Nichols algebras. Let H be a Hopf algebra. We shall say *braided* Hopf algebra for a Hopf algebra in the braided tensor category ${}_H^H \mathcal{YD}$. Given

$V \in {}^H_H\mathcal{YD}$, the *Nichols algebra* of V is the braided graded Hopf algebra $\mathcal{B}(V) = \bigoplus_{n \geq 0} \mathcal{B}^n(V)$ satisfying the following conditions:

- $\mathcal{B}^0(V) \simeq \mathbb{k}$, $\mathcal{B}^1(V) \simeq V$ as Yetter-Drinfeld modules over H ;
- $\mathcal{B}^1(V) = \mathcal{P}(\mathcal{B}(V))$, the set of primitives elements of $\mathcal{B}(V)$;
- $\mathcal{B}(V)$ is generated as an algebra by $\mathcal{B}^1(V)$.

The Nichols algebra of V exists and is unique up to isomorphism. We sketch a way to construct $\mathcal{B}(V)$ and prove its unicity, see [AS2]. Note that the tensor algebra $T(V)$ admits a unique structure of graded braided Hopf algebra in ${}^H_H\mathcal{YD}$ such that $V \subseteq \mathcal{P}(V)$. Consider the class \mathfrak{S} of all the homogeneous two-sided Hopf ideals $I \subseteq T(V)$ such that I is generated by homogeneous elements of degree ≥ 2 and is a Yetter-Drinfeld submodule of $T(V)$. Then $\mathcal{B}(V)$ is the quotient of $T(V)$ by the maximal element $I(V)$ of \mathfrak{S} . Thus, the canonical projection $\pi : T(V) \rightarrow \mathcal{B}(V)$ is a Hopf algebra surjection in ${}^H_H\mathcal{YD}$.

Braided vector spaces of diagonal type with finite-dimensional Nichols algebra were classified in [He1]; the explicit defining relations were given in [Ang3], using the results of [Ang2]. Presently we understand that the list of braidings of diagonal type with finite-dimensional Nichols algebra given in [He1] is divided into three parts:

- (1) Standard type [Ang1], comprising Cartan type [AS1];
- (2) Super type [AAY];
- (3) Unidentified type [Ang4].

2. SPHERICAL HOPF ALGEBRAS

2.1. Spherical Hopf algebras. The notion of *spherical Hopf algebra* was introduced in [BaW1]: this is a pair (H, ω) , where H is a Hopf algebra and $\omega \in G(H)$ such that

$$(2.1) \quad \mathcal{S}^2(x) = \omega x \omega^{-1}, \quad x \in H,$$

$$(2.2) \quad \mathrm{tr}_V(\vartheta \omega) = \mathrm{tr}_V(\vartheta \omega^{-1}), \quad \vartheta \in \mathrm{End}_H(V),$$

for all $V \in \mathrm{Rep} H$. We say that $\omega \in G(H)$ is a *pivot* when it satisfies (2.1); pairs (H, ω) with ω a pivot are called *pivotal Hopf algebras*. The pivot is not unique but it is determined up to multiplication by an element in the group $G(H) \cap Z(H)$. A *spherical element* is a pivot that fulfills (2.2).

The implementation of the square of the antipode by conjugation by a group-like, condition (2.1), is easy to verify. For instance, ω should belong to the center of $G(H)$; thus, if this group is centerless, then (2.1) does not hold in H . Further, the failure of (2.1) is not difficult to remedy by adjoining a group-like element [So, Section 2]. Namely, given a Hopf algebra H , consider a cyclic group Γ of order $\mathrm{ord} \mathcal{S}^2$ with a generator g ; let g act on H as \mathcal{S}^2 . Then the corresponding smash product $E(H) := H \# \mathbb{k}\Gamma$ is a Hopf algebra where (2.1) holds².

²If H has finite dimension $n \in \mathbb{N}$, then the order of \mathcal{S}^2 is finite; in fact, it divides $2n$, by Radford's formula on \mathcal{S}^4 and the Nichols-Zöller Theorem.

Condition (2.2) is less apparent. If H is a finite-dimensional Hopf algebra and $\omega \in H$ a pivot, then (2.2) holds in the following instances:

- ω is an involution.
- There exists a Hopf subalgebra K of H such that $\omega \in K$ and (K, ω) is spherical— since $\text{End}_H(V) \subset \text{End}_K(V)$.
- H is ribbon, see Subsection 2.7.
- All finite-dimensional H -modules are naturally self-dual.

Proof. By hypothesis, there exists a natural isomorphism $F : \text{id} \rightarrow *$. Let $V \in \text{Rep } H$ and $\vartheta \in \text{End}_H(V)$. Then

$$\begin{aligned} \text{tr}(\vartheta\omega) &= \sum_i \langle \alpha_i, \vartheta\omega v_i \rangle = \sum_i \langle \vartheta^* \alpha_i, \omega v_i \rangle = \sum_i \langle F_V \vartheta F_V^{-1} \alpha_i, \omega v_i \rangle \\ &= \sum_i \langle F_V \vartheta \omega^{-1} F_V^{-1} \alpha_i, v_i \rangle = \text{tr}(F_V \vartheta \omega^{-1} F_V^{-1}) = \text{tr}(\vartheta \omega^{-1}). \end{aligned}$$

Here $\{v_i\}$ and $\{\alpha_i\}$ are dual basis of V and V^* respectively. \square

Proposition 2.1. *A pivotal Hopf algebra (H, ω) is spherical if and only if (2.2) holds for all $S \in \text{Irr } H$.*

Proof. The Proposition follows from the following two claims.

Claim 1. If (2.2) holds for $M_1, M_2 \in \text{Rep } H$, then it holds for $M_1 \oplus M_2$.

Indeed, let $h \in \text{End}_H(M_1 \oplus M_2)$. Let $\pi_j : M \rightarrow M_j$, $\iota_i : M_i \rightarrow M$ be the projection and the inclusion, for $1 \leq i, j \leq 2$. Then $h = \sum_{1 \leq i, j \leq 2} h_{ij}$, where $h_{ij} = \pi_i \circ h \circ \iota_j \in \text{Hom}_H(M_j, M_i)$. In particular, $h_{ij}\omega = (\overline{h\omega})_{ij}$ as linear maps. Now, we have that $\text{tr}_M(h) = \text{tr}_{M_1}(h_{11}) + \text{tr}_{M_2}(h_{22})$ and thus

$$\begin{aligned} \text{tr}_M(h\omega) &= \text{tr}_{M_1}(h\omega)_{11} + \text{tr}_{M_2}(h\omega)_{22} = \text{tr}_{M_1}(h_{11}\omega) + \text{tr}_{M_2}(h_{22}\omega) \\ &= \text{tr}_{M_1}(h_{11}\omega^{-1}) + \text{tr}_{M_2}(h_{22}\omega^{-1}) = \text{tr}_{M_1}(h\omega^{-1})_{11} + \text{tr}_{M_2}(h\omega^{-1})_{22} \\ &= \text{tr}_M(h\omega^{-1}). \end{aligned}$$

Claim 2. If (2.2) holds for every semisimple H -module, then H is spherical.

Let $M \in \text{Rep } H$ and let $M_0 \subset M_1 \subset \cdots \subset M_k = M$ be the *Loewy filtration* of M , that is $M_0 = \text{Soc } M$, $M_{i+1}/M_i \simeq \text{Soc}(M/M_i)$, $i = 0, \dots, k-1$. In particular, M_{i+1}/M_i is semisimple. We prove the claim by induction on the Loewy length k of M . The case $k = 0$ is the hypothesis. Assume $k > 0$; set $S = \text{Soc } M$ and consider the exact sequence $0 \rightarrow S \rightarrow M \rightarrow M/S \rightarrow 0$. Hence the Loewy length of $\widetilde{M} = M/S$ is $k-1$ and thus (2.2) holds for it. Also, (2.2) holds for S by hypothesis. Let $f \in \text{End}_H(M)$, then $f(S) \subseteq S$ and thus f induces $f_1 = f|_S \in \text{End}_H(S)$ by restriction and factorizes through $f_2 \in \text{End}_H(\widetilde{M})$. Therefore, we can choose a basis of S and complete it to a basis of M in such a way that f in this new basis is represented by $[f] = \begin{bmatrix} [f_1] & * \\ 0 & [f_2] \end{bmatrix}$. Also, as ω preserves S and f is an H -morphism, it follows that $[f\omega] = \begin{bmatrix} [f_1\omega] & * \\ 0 & [f_2\omega] \end{bmatrix}$ and thus $\text{tr}_M(f\omega) = \text{tr}_M(f\omega^{-1})$. \square

Example 2.2. Let H be a basic Hopf algebra, i. e. all finite-dimensional simple modules have dimension 1; when H itself is finite-dimensional, this amounts to the dual of H being pointed. If $\omega \in G(H)$ is a pivot, then H is spherical if and only if $\chi(\omega) \in \{\pm 1\}$ for all $\chi \in \text{Alg}(H, \mathbb{k})$. Assume that H is finite-dimensional; then H is spherical if and only if ω is involutive. For, $\omega - \omega^{-1} \in \bigcap_{\chi \in \text{Alg}(H, \mathbb{k})} \text{Ker } \chi = \text{Rad } H$, and $\text{Rad } H \cap \mathbb{k}[\omega] = 0$.

2.2. Spherical Categories. A monoidal rigid category \mathcal{C} is pivotal when X^{**} is monoidally isomorphic to X [FY1]; this implies that the left and right dualities coincide. For instance, if H is a Hopf algebra and $\omega \in G(H)$ is a pivot, then $\text{Rep } H$ is pivotal [BaW1, Proposition 3.6]. In a pivotal category \mathcal{C} , there are left and right traces $\text{tr}_L, \text{tr}_R : \text{End}(X) \rightarrow \text{End}(\mathbf{1})$, for any $X \in \mathcal{C}$. If $\mathcal{C} = \text{Rep } H$ and $V \in \text{Rep } H$, then these traces are defined by

$$(2.3) \quad \text{tr}_L(\vartheta) = \text{tr}_V(\vartheta\omega), \quad \text{tr}_R(\vartheta) = \text{tr}_V(\vartheta\omega^{-1}), \quad \vartheta \in \text{End}_H(V).$$

A spherical category is a pivotal one where the left and right traces coincide. Thus, $\text{Rep } H$ is a spherical category, whenever H is a spherical Hopf algebra.

Remark 2.3. If \mathcal{D} is a rigid monoidal (full) subcategory of a spherical category \mathcal{C} , then \mathcal{D} is also spherical.

2.3. Quantum dimensions. The *quantum dimension* of an object X in a spherical category \mathcal{C} is given by $\text{qdim } X := \text{tr}_L(\text{id}_X)$. In particular, if H is a spherical Hopf algebra, then

$$(2.4) \quad \text{qdim } M = \text{tr}_M(\omega) = \text{tr}_M(\omega^{-1}), \quad M \in \text{Rep } H.$$

2.3.1. If \mathcal{C} is a spherical tensor category, then $\text{End}(\mathbf{1}) = \mathbb{k}$, and the function $V \mapsto \text{qdim } V$ is a character of the Grothendieck ring of \mathcal{C} . In fact, this map is additive on exact sequences, as in the proof of Proposition 2.1; also

$$(2.5) \quad \text{qdim } V \otimes W = \text{qdim } V \text{ qdim } W, \quad V, W \in \mathcal{C}.$$

In consequence, the quantum dimension of any object in a *finite* spherical tensor category \mathcal{C} is an algebraic integer in \mathbb{k} , see [EGNO, Corollary 1.38.6].

2.3.2. Let H be a Hopf algebra. Given $L \in \text{Irr } H$, $M \in \text{Rep } H$, we set $(M : L) = \text{multiplicity of } L \text{ in } M$ (i. e. the number of times that L appears as a Jordan-Hölder factor of M). Assume that (H, ω) is spherical. Let $M \in \text{Rep } H$. Then

$$(2.6) \quad \text{qdim } M = \sum_{L \in \text{Irr } H} (M : L) \text{qdim } L.$$

Here is a way to compute the quantum dimension of M : consider the decomposition $M = \bigoplus_{\rho \in \text{Irr } G(H)} M_\rho$ into isotypical components of the restriction of M to $G(H)$. Since $\omega \in Z(G(H))$, it acts by a scalar z_ρ on the $G(H)$ -module affording $\rho \in \text{Irr } G(H)$. Hence

$$(2.7) \quad \text{qdim } M = \sum_{\rho \in \text{Irr } G(H)} z_\rho \dim M_\rho.$$

See [CW1, CW2] for a Verlinde formula and other information on the computation of the quantum dimension in terms of the Grothendieck ring.

2.3.3. If H is a non-semisimple spherical Hopf algebra, then $\text{qdim } M = 0$ for any $M \in \text{Rep } H$ projective [BaW2, Proposition 6.10]. More generally, the following result appears in the proofs of [EO, 2.16], [EGNO, 1.53.1].

Proposition 2.4. *Let \mathcal{C} be a non-semisimple pivotal tensor category. Then $\text{qdim } P = 0$ for any projective object P .* \square

2.4. The non-degenerate quotient. Let \mathcal{C} be an additive \mathbb{k} -linear spherical category with $\text{End}(\mathbf{1}) \simeq \mathbb{k}$. For any two objects X, Y in \mathcal{C} there is a bilinear pairing

$$\Theta : \text{Hom}_{\mathcal{C}}(X, Y) \times \text{Hom}_{\mathcal{C}}(Y, X) \rightarrow \mathbb{k}, \quad \Theta(fg) = \text{tr}_L(fg) = \text{tr}_R(gf);$$

\mathcal{C} is non-degenerate if Θ is, for any X, Y . By [BaW1, Theorem 2.9], see also [T], any additive spherical category \mathcal{C} gives rise a factor category $\underline{\mathcal{C}}$, with the same objects³ as \mathcal{C} and morphisms $\text{Hom}_{\underline{\mathcal{C}}}(X, Y) := \text{Hom}_{\mathcal{C}}(X, Y) / \mathcal{J}(X, Y)$, $X, Y \in \mathcal{C}$, where

$$(2.8) \quad \mathcal{J}(X, Y) = \{f \in \text{Hom}_{\mathcal{C}}(X, Y) : \text{tr}_L(fg) = 0, \forall g \in \text{Hom}_{\mathcal{C}}(Y, X)\}.$$

The category $\underline{\mathcal{C}}$ is an additive non-degenerate spherical category, but it is not necessarily abelian, even if \mathcal{C} is abelian. Clearly, the quantum dimensions in $\text{Rep } H$ and $\underline{\text{Rep}} H$ are the same. See [Kn] for a general formalism of tensor ideals, that encompasses the construction above.

We now give more information on $\underline{\text{Rep}} H$ following [BaW1, Proposition 3.8]. Let us first point out some precisions on the terminology used in the literature on additive categories, see e. g. [Har] and references therein. We shall stick to this terminology in what follows. Let \mathcal{C} be an additive k -linear category, where k is an arbitrary field.

- \mathcal{C} is *semisimple* if the algebras $\text{End}(X)$ are semisimple for all $X \in \mathcal{C}$.
- An object X is *minimal* if any monomorphism $Y \rightarrow X$ is either 0 or an isomorphism; hence $\text{End}(X)$ is a division algebra over k .
- \mathcal{C} is *completely reducible* if every object is a direct sum of minimal ones.

Beware that in more recent literature in topology (where it is assumed that $k = \mathbb{k}$), an object S is said to be *simple* when $\text{End}(S) \simeq \mathbb{k}$; and ‘ \mathcal{C} is *semisimple*’ means that every object in \mathcal{C} is a direct sum of simple ones. For instance, if $\mathcal{C} = A\text{-mod}$, A an algebra, then a minimal object in \mathcal{C} is just a simple A -module; then $\text{End}(S) \cong \mathbb{k}$ by the Schur Lemma. But it is well-known that the converse is not true.

Example 2.5. Let $H = \mathbb{k}\langle x, g | x^2, g^2 - 1, gx + xg \rangle$ be the Sweedler Hopf algebra. H has two simple modules, both one dimensional, namely the trivial

³This is a bit misleading, as non-isomorphic objects in \mathcal{C} may become isomorphic in $\underline{\mathcal{C}}$.

V^+ and V^- , where g acts as -1 . Consider the non-trivial extension $V^\pm \in \text{Ext}_H^1(V^-, V^+)$, that is $V^\pm = \mathbb{k}v \oplus \mathbb{k}w$ with action

$$g \cdot v = v, \quad g \cdot w = -w, \quad x \cdot v = w, \quad x \cdot w = 0.$$

Then V^\pm is not simple and $\text{End}_H(V^\pm) \cong \mathbb{k}$. It is easy to see that the indecomposable H -modules are V^+ , V^- , V^\pm and $V^\mp := (V^\pm)^*$; hence $\underline{\text{Rep}} H \simeq \text{Rep } \mathbb{Z}/2$ by Step 2 of the proof of Theorem 2.7 below. Also, notice that in $\text{Rep } H$ it is not true that all endomorphism algebras are semisimple (just take the regular representation); and, related to this, that $\text{Hom}_H(V^\mp, V^\pm) \neq 0$.

However, the converse above is true when the category is semisimple.

Remark 2.6. [Har, 1.1, 1.3] Let \mathcal{C} be a semisimple additive k -linear category, where k is an arbitrary field.

- (a) If $\alpha : V \rightarrow W$ is not zero, then there exists $\beta, \gamma : W \rightarrow V$ such that $\beta\alpha \neq 0, \alpha\gamma \neq 0$. If $\text{Hom}_{\mathcal{C}}(V, W) = 0$, then $\text{Hom}_{\mathcal{C}}(W, V) = 0$.
- (b) If $V \in \mathcal{C}$ and $\text{End}(V)$ is a division ring, then V is minimal.
- (c) If $V, W \in \mathcal{C}$ are minimal and non-isomorphic, then $\text{Hom}_{\mathcal{C}}(V, W) = 0$.

Proof. (a) Assume that $\text{Hom}_{\mathcal{C}}(W, V)\alpha = 0$. Set $U = V \oplus W$; then $\text{End}_{\mathcal{C}} U\alpha$ is a nilpotent left ideal of $\text{End}_{\mathcal{C}} U$, hence it is 0.

(b) Let $W \neq 0$ and $f \in \text{Hom}(W, V)$ be a monomorphism. Since $f \neq 0$, $\text{Hom}(V, W) \neq 0$ by (a). Since $\text{End}(V)$ is a division ring, the map $f \circ - : \text{Hom}(V, W) \rightarrow \text{End}(V)$ is surjective. Therefore there exists $g \in \text{Hom}(V, W)$ such that $f \circ g = \text{id}_V$. On the other hand, the map $\text{End}(W) \rightarrow \text{Hom}(W, V)$, $\tilde{h} \mapsto f \circ \tilde{h}$ is injective. Hence $g \circ f = \text{id}_W$ since $f \circ (g \circ f) = f \circ \text{id}_W$. Thus $W \simeq V$. (c) follows from (a) at once. \square

If H is a spherical Hopf algebra, then we denote by $\text{Indec}_q H$ the set of isomorphism classes of indecomposable finite-dimensional H -modules with non-zero quantum dimension.

Theorem 2.7. [BaW1, 3.8] *Let H be a spherical Hopf algebra with pivot ω . Then the non-degenerate quotient $\underline{\text{Rep}} H$ is a completely reducible spherical tensor category, and $\text{Irr } \underline{\text{Rep}} H$ is in bijective correspondence with $\text{Indec}_q H$.*

By Remark 2.6, ‘completely reducible’ becomes what is called semisimple in the recent literature.

Proof. The crucial step is to show that $\underline{\text{Rep}} H$ is semisimple.

Step 1. [BaW1, 3.7] The algebra $\text{End}_{\underline{\text{Rep}} H}(X)$ is semisimple for any X in $\underline{\text{Rep}} H$.

Indeed, the Jacobson radical J of $\text{End}_H(X)$ is contained in $\mathcal{J}(X, X)$. For, if $\vartheta \in J$, then $\vartheta\omega$ is nilpotent, hence $\text{tr}_L(\vartheta) = \text{tr}_V(\vartheta\omega) = 0$.

As a consequence of Step 1 and Remark 2.6, $X \in \underline{\text{Rep}} H$ is minimal if and only if $\text{End}_{\underline{\text{Rep}} H}(X) \simeq \mathbb{k}$, that is if and only if it is simple. Also, $\text{Hom}_{\underline{\text{Rep}} H}(S, T) = 0$, for $S, T \in \underline{\text{Rep}} H$ simple non-isomorphic.

Step 2. V is a simple object in $\underline{\text{Rep}} H$ iff there exists $W \in \text{Rep } H$ indecomposable with $\text{qdim } W \neq 0$ which is isomorphic to V in $\underline{\text{Rep}} H$.

Assume that $W \in \text{Rep } H$ is indecomposable. If $f \in \text{End}_H(W)$, then f is either bijective or nilpotent by the Fitting Lemma. If also $\text{qdim } W \neq 0$, then $\text{End}_{\underline{\text{Rep}} H}(W)$ is a finite dimensional division algebra over \mathbb{k} , necessarily isomorphic to \mathbb{k} . Now, assume that V is a simple object in $\underline{\text{Rep}} H$. Let $\pi \in \text{End}_H(V)$ be a lifting of $\text{id}_V = 1 \in \text{End}_{\underline{\text{Rep}} H}(V) \simeq \mathbb{k}$. We can choose π to be a primitive idempotent. Then the image W of π is indecomposable and $\pi|_W$ induces an isomorphism between W and V in $\underline{\text{Rep}} H$. Again by the Fitting Lemma, $\text{End}_H(W) \simeq \mathbb{k}\pi|_W \oplus \text{Rad } \text{End}_H(W)$. Hence $\text{qdim } W \neq 0$ since π is a lifting of $\text{id}_V \in \text{End}_{\underline{\text{Rep}} H}(V)$.

Step 3. Let $V, W \in \text{Rep } H$ indecomposable with $\text{qdim } V \neq 0$, $\text{qdim } W \neq 0$, which are isomorphic in $\underline{\text{Rep}} H$. Then $V \simeq W$ in $\text{Rep } H$.

Let $f \in \text{Hom}_H(V, W)$ and $g \in \text{Hom}_H(W, V)$ such that $gf = \text{id}_V$ in $\underline{\text{Rep}} H$; that is

$$(2.9) \quad \text{tr}_V(\vartheta(gf - \text{id})\omega) = 0 \quad \text{for every } \vartheta \in \text{End}_H(V).$$

Since V is indecomposable, gf is invertible in $\text{End}(V)$, or otherwise (2.9) would fail for $\vartheta = \text{id}$. Thus g is surjective and f is injective. But W is also indecomposable, hence f is surjective and g is injective, and both are isomorphisms.

To finish the proof of the statement about the irreducibles, observe that an indecomposable $U \in \text{Rep } H$ with $\text{qdim } U = 0$ satisfies $U \simeq 0$ in $\underline{\text{Rep}} H$. Since any $M \in \text{Rep } H$ is a direct sum of indecomposables, we see that M is isomorphic in $\underline{\text{Rep}} H$ to a direct sum of indecomposables with non-zero quantum dimension.

Finally observe that the additive \mathbb{k} -linear category $\underline{\text{Rep}} H$, being isomorphic to a direct sum of copies of $\text{Vec } \mathbb{k}$, is abelian. \square

Here is a consequence of the Theorem: let $\iota : V \hookrightarrow W$ be a proper inclusion of indecomposable H -modules with $\text{qdim } V \neq 0$, $\text{qdim } W \neq 0$. Then $\iota \in \mathcal{J}(V, W)$.

Remark 2.8. From Theorem 2.7 we see the relation between the constructions of [BaW1] and [GK]. For, let $\mathcal{C} = \text{Rep } H$ and let \mathcal{C}^0 , resp. \mathcal{C}^\perp , be the full subcategory whose objects are direct sums of indecomposables with quantum dimension $\neq 0$, resp. 0. Then $\underline{\text{Rep}} H$ is the quotient of \mathcal{C} by \mathcal{C}^\perp as described in [GK, Section 1].

Even when H is finite-dimensional, $\text{Irr } \underline{\text{Rep}} H$ is not necessarily finite. It is then natural to look at suitable subcategories of $\text{Rep } H$ that give rise to finite tensor subcategories of $\underline{\text{Rep}} H$. A possibility is tilting modules, that proved to be very fruitful in the case of quantum groups at roots of one. We shall discuss this matter in Section 3.

2.5. Pointed or copointed pivotal Hopf algebras.

2.5.1. Let H be a pointed Hopf algebra and set $G = G(H)$. We assume that H is generated by group-like and skew-primitive elements. For H finite-dimensional, it was conjectured that this is always the case [AS1, Conjecture 1.4]. So far this is true in all known cases, see [Ang3, AGI] and references therein. As explained in Subsection 1.3, there exist $g_1, \dots, g_\theta \in G$ and $\rho_i \in \text{Irr } C_G(g)$, $1 \leq i \leq \theta$, such that the infinitesimal braiding of H is

$$M(\mathcal{O}_{g_1}, \rho_1) \oplus \dots \oplus M(\mathcal{O}_{g_\theta}, \rho_\theta).$$

Lemma 2.9. *Let $\omega \in G$. Then the following are equivalent:*

- (a) ω is a pivot.
- (b) $\omega \in Z(G)$ and $\rho_i(\omega) = \rho_i(g_i)^{-1}$, $1 \leq i \leq \theta$.

Proof. There exist $x_1, \dots, x_\theta \in H$ such that $\Delta(x_i) = x_i \otimes 1 + g_i \otimes x_i$, $1 \leq i \leq \theta$, and H is generated by x_1, \dots, x_θ and G as an algebra. Now $\mathcal{S}^2(x_i) = g_i^{-1} x_i g_i = \rho_i(g_i^{-1}) x_i$. \square

2.5.2. Let G be a finite group and δ_g be the characteristic function of the subset $\{g\}$ of G . If $M \in {}_{\mathbb{k}G}\mathcal{M}$, then $M = \bigoplus_{g \in \text{Supp } M} M[g]$ where

$$M[g] := \delta_g \cdot M \text{ and } \text{Supp } M := \{g \in G : M[g] \neq 0\}.$$

Lemma 2.10. *Let H be a finite-dimensional copointed Hopf algebra over \mathbb{k}^G and $\omega = \sum_{g \in G} \omega(g) \delta_g \in G(\mathbb{k}^G)$. The following are equivalent:*

- (a) ω is a pivot.
- (b) $\mathcal{S}^2(x) = \omega(g)x$ for all $x \in H[g]$, $g \in G$.

Proof. Consider H as a \mathbb{k}^G -module via the adjoint action; then $\delta_t x = x \delta_{g^{-1}t}$ for $x \in H[g]$, $g, t \in G$ [AV1, 3.1 (b)]. Hence $\omega x \omega^{-1} = \omega(g)x$. \square

2.6. Spherical Hopf algebras with involutory pivot. There are many examples of Hopf algebras with involutory pivot.

2.6.1. Let H be a finite-dimensional pointed Hopf algebra with $G(H)$ abelian. Then its infinitesimal braiding V is a braided vector space of diagonal type with matrix $(q_{ij})_{1 \leq i, j \leq \theta}$, $\theta = \dim V$. Assume that $q_{ii} = -1$, $1 \leq i \leq \theta$. The list of all braided vector spaces with this property and such that the associated Nichols algebra is finite-dimensional can be easily extracted from the main result of [He1]. We apply Lemma 2.9 because H is generated by group-like and skew-primitive elements [Ang3]. Hence, if V belongs to this list, then H is a spherical Hopf algebra with involutory pivot, eventually adjoining a group-like if necessary. The argument also works when $G(H)$ is not abelian but the infinitesimal braiding is a direct sum of one-dimensional Yetter-Drinfeld modules (one often says that the infinitesimal braiding *comes from the abelian case*).

2.6.2. Let H be a finite-dimensional pointed Hopf algebra with $G(H)$ not abelian and such that the infinitesimal braiding does not come from the abelian case. In all the examples of such infinitesimal braidings that are known⁴, we may apply Lemma 2.9 because H is generated by group-like and skew-primitive elements. Also, in all examples except one in [HLV], the scalar in Lemma 2.9 (b) is -1. Thus H is a spherical Hopf algebra with involutory pivot, eventually adjoining a group-like if necessary.

2.6.3. Let H be a finite-dimensional copointed Hopf algebra over \mathbb{k}^G , with G not abelian. Lemma 2.10 makes it easy to check whether H has an involutory pivot. For instance, the Hopf algebras $\mathcal{A}_{[\mathbf{a}]}$, for $\mathbf{a} \in \mathfrak{A}_3$, introduced in [AV1], have an involutory pivot. These Hopf algebras are liftings of $\mathcal{B}(V_3) \# \mathbb{k}^{\mathbb{S}_3}$, so they have dimension 72; but they are not quasi-triangular [AV1].

Also, $\text{Irr Rep } \mathcal{A}_{[0]}$ is infinite. Namely, $\mathcal{A}_{[0]} \simeq \mathcal{B}(V_3) \# \mathbb{k}^{\mathbb{S}_3}$ and for $\lambda \in \mathbb{k}$ we define the $\mathcal{A}_{[0]}$ -module $M_\lambda = \langle m_g : e \neq g \in \mathbb{S}_3 \rangle$ by

$$\delta_h \cdot m_g = \delta_h(g) m_g \quad \text{and} \quad x_{(ij)} \cdot m_g = \begin{cases} 0 & \text{if } \text{sgn } g = -1, \\ \lambda_{(ij),g} m_{(ij)g} & \text{if } \text{sgn } g = 1, \end{cases}$$

where $\lambda_{(ij),g} = 1$ except to $\lambda_{(12),(123)} = \lambda$. Then $M \in \text{Indec}_q \mathcal{A}_{[0]}$ with $\text{qdim } M_\lambda = -1$ for all $\lambda \in \mathbb{k}$ and $M_\lambda \simeq M_\mu$ iff $\lambda = \mu$. Analogously, we can define $N_\lambda \in \text{Indec}_q \mathcal{A}_{[0]}$ with basis $\langle n_g : (12) \neq g \in \mathbb{S}_3 \rangle$, $\text{qdim } N_\lambda = 1$ and which are mutually not isomorphic.

Remark 2.11. The dual of $\mathcal{A}_{[0]}$ is $\mathcal{B}(V_3) \# \mathbb{k}^{\mathbb{S}_3}$, which is not pivotal because \mathbb{S}_3 is centerless. Compare with the main result of [P], where it is shown that the dual of a semisimple spherical Hopf algebra is again spherical. In fact, the dual of $\mathcal{A}_{[\mathbf{a}]}$ is not pivotal for any $\mathbf{a} \in \mathfrak{A}_3$. If \mathbf{a} is generic, then $(\mathcal{A}_{[\mathbf{a}]})^*$ has no non-trivial group-likes [AV2, Theorem 1]. If \mathbf{a} is sub-generic, then the unique non-trivial group-like $\zeta_{(12)}$ of $(\mathcal{A}_{[\mathbf{a}]})^*$ is not a pivot, see [AV2, Lemma 8] for notations. Namely, if $g \neq (12), e$, then

$$\zeta_{(12)} \rightharpoonup \delta_g \leftarrow \zeta_{(12)} = \sum_{t,s \in \mathbb{S}_3} \delta_s((12)) \delta_{s^{-1}t} \delta_{t^{-1}g}((12)) = \delta_{(12)g(12)} \neq \mathcal{S}^2(\delta_g).$$

2.6.4. Let H be a spherical Hopf algebra with involutory pivot ω . Then

- The quantum dimensions are integers.
- If χ is a representation of dimension one, then $\text{qdim } \mathbb{k}_\chi = \chi(\omega)$.
- If H is not semisimple, then at least one module has negative quantum dimension.
- Assume that there exists $L \in \text{Irr } H$ such that $\text{qdim } L' > 0$ for all $L' \in \text{Irr } H$, $L' \neq L$. Then $\text{qdim } L < 0$.

Proposition 2.12. *Let \mathcal{C} be a fusion subcategory of $\text{Rep } H$, where H is a spherical Hopf algebra with involutory pivot. Then there exists a semisimple quasi-Hopf algebra K such that $\mathcal{C} \simeq \text{Rep } K$ as fusion categories.*

⁴The list of all known examples is in <http://mate.dm.uba.ar/~matiasg/zoo.html>, see also [GHV, Table 1], except for one example discovered later [HLV, Proposition 36].

Proof. The quantum dimensions are integers, because the pivot is involutory, and positive by [ENO, Corollary 2.10]; here we use that \mathcal{C} is spherical, see Remark 2.3. Then [ENO, Theorem 8.33] applies. Indeed, the Perron-Frobenius and quantum dimensions here coincide, see e. g. the proof of [ENO, Proposition 8.23]. \square

We are inclined to believe, because of some computations in examples, that the quasi-Hopf algebra K in the statement is actually a Hopf algebra quotient of H .

2.7. Ribbon Hopf algebras. This is a distinguished class of spherical Hopf algebras. Let (H, \mathcal{R}) be a quasi-triangular Hopf algebra [Dr]. We denote the universal matrix as $\mathcal{R} = \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)}$. The *Drinfeld element* is

$$(2.10) \quad \mathbf{u} = \mathcal{S}(\mathcal{R}^{(2)})\mathcal{R}^{(1)}.$$

Let $Q = \mathcal{R}_{21}\mathcal{R}$. The Drinfeld element is invertible and satisfies

$$(2.11) \quad \Delta(\mathbf{u}) = Q^{-1}(\mathbf{u} \otimes \mathbf{u}) = (\mathbf{u} \otimes \mathbf{u})Q^{-1}, \quad \mathbf{u}^{-1} = \mathcal{R}^{(2)}\mathcal{S}^2(\mathcal{R}^{(1)}),$$

$$(2.12) \quad g := \mathbf{u}\mathcal{S}(\mathbf{u})^{-1} = \mathcal{S}(\mathbf{u})^{-1}\mathbf{u} \in G(H), \quad \mathbf{u}\mathcal{S}(\mathbf{u}) \in Z(H),$$

$$(2.13) \quad \mathcal{S}^2(h) = \mathbf{u}h\mathbf{u}^{-1}, \quad \mathcal{S}^4(h) = ghg^{-1},$$

for any $h \in H$.

Definition 2.13. [RT1] A quasi-triangular Hopf algebra (H, \mathcal{R}) is *ribbon* if there exists $\mathbf{v} \in Z(H)$, called the *ribbon element*, such that

$$(2.14) \quad \mathbf{v}^2 = \mathbf{u}\mathcal{S}(\mathbf{u}), \quad \mathcal{S}(\mathbf{v}) = \mathbf{v}, \quad \Delta(\mathbf{v}) = Q^{-1}(\mathbf{v} \otimes \mathbf{v}).$$

The ribbon element is not unique but it is determined up to multiplication by an element in $\{g \in G(H) \cap Z(H) : g^2 = 1\}$.

Let H be a ribbon Hopf algebra. It follows easily that $\omega = \mathbf{u}\mathbf{v}^{-1} \in G(H)$ and $\mathcal{S}^2(h) = \omega h \omega^{-1}$ for all $h \in H$; that is, ω is a pivot. Actually, H is spherical [BaW1, Example 3.2]. In fact, the concept of *quantum trace* is defined in any ribbon category using the braiding, see for example [K, XIV.4.1]. By [K, XIV.6.4], the *quantum trace* of $\text{Rep } H$ coincides with tr_L , cf. (2.3). Moreover, [K, XIV.4.2 (c)] asserts that $\text{tr}_L = \text{tr}_R$.

There are quasi-triangular Hopf algebras that are not ribbon, but the failure is not difficult to remedy by adjoining a group-like element [RT1, Theorem 3.4]. Namely, given a quasi-triangular Hopf algebra (H, \mathcal{R}) , let $\tilde{H} = H \oplus H\mathbf{v}$, where \mathbf{v} is a formal element not in H . Then \tilde{H} is a Hopf algebra with product, coproduct, antipode and counit defined for $x, x', y, y' \in H$ by

$$(2.15) \quad (x + y\mathbf{v}) \cdot (x' + y'\mathbf{v}) = (xx' + yy'\mathbf{u}\mathcal{S}(\mathbf{u})) + (xy' + yx')\mathbf{v},$$

$$(2.16) \quad \Delta(x + y\mathbf{v}) = \Delta(x) + \Delta(y)Q^{-1}(\mathbf{v} \otimes \mathbf{v}),$$

$$(2.17) \quad \mathcal{S}(x + y\mathbf{v}) = \mathcal{S}(x) + \mathcal{S}(y)\mathbf{v}, \quad \varepsilon(x + y\mathbf{v}) = \varepsilon(x) + \varepsilon(y).$$

Clearly, H becomes a Hopf subalgebra of \widetilde{H} ; it can be shown then that \mathcal{R} is a universal R -matrix for \widetilde{H} and that \mathbf{v} is a ribbon element for $(\widetilde{H}, \mathcal{R})$. See [RT1, Theorem 3.4]. We shall say that \widetilde{H} is the *ribbon extension* of (H, \mathcal{R}) .

Remark 2.14. (Y. Sommerhäuser, private communication). The ribbon extension fits into an exact sequence of Hopf algebras $H \hookrightarrow \widetilde{H} \twoheadrightarrow \mathbb{k}[\mathbb{Z}/2]$, which is cleft. Namely, let ξ be the generator of $\mathbb{Z}/2$ and define

$$\xi \rightharpoonup x = \mathcal{S}^2(x), \quad x \in H, \quad \sigma(\xi^i \otimes \xi^j) = g^{ij}, \quad i, j \in \{0, 1\}.$$

Then the crossed product defined by this action and cocycle together with the tensor product of coalgebras is a Hopf algebra $H \#_{\sigma} \mathbb{k}[\mathbb{Z}/2]$, see for instance [AD]. Now the map $\psi : H \#_{\sigma} \mathbb{k}[\mathbb{Z}/2] \rightarrow \widetilde{H}$, $\psi(x \# \xi^i) = x(\mathcal{S}(u^{-1})\mathbf{v})^i$, is an isomorphism of Hopf algebras.

In conclusion, any finite-dimensional Hopf algebra H gives rise to a ribbon Hopf algebra, namely the ribbon extension of its Drinfeld double:

$$H \rightsquigarrow D(H) \rightsquigarrow \widetilde{D(H)}.$$

Remark 2.15. A natural question is whether the Drinfeld double itself is ribbon; this was addressed in [KaR], where the following results were obtained. Let H be a finite-dimensional Hopf algebra, $g \in G(H)$ and $\alpha \in G(H^*)$ be the distinguished group-likes⁵. The celebrated Radford's formula for the fourth power of the antipode [R] states that

$$(2.18) \quad \mathcal{S}^4(h) = g(\alpha \rightharpoonup h \leftarrow \alpha^{-1})g^{-1}, \quad h \in H.$$

Here $\rightharpoonup, \leftarrow$ are the transposes of the regular actions.

(a) [KaR, Theorem 2] Suppose that (H, \mathcal{R}) is quasi-triangular and that $G(H)$ has odd order. Then (H, \mathcal{R}) admits a (necessarily unique) ribbon element if and only if \mathcal{S}^2 has odd order.

(b) [KaR, Theorem 3] $(D(H), \mathcal{R})$ admits a ribbon element if and only if there exist $\ell \in G(H)$ and $\beta \in G(H^*)$ such that

$$(2.19) \quad \ell^2 = g, \quad \beta^2 = \alpha, \quad \mathcal{S}^2(h) = \ell(\beta \rightharpoonup h \leftarrow \beta^{-1})\ell^{-1}, \quad h \in H.$$

2.8. Cospherical Hopf algebras. It is natural to look at the notions that insure that the category of comodules of a Hopf algebra is pivotal or spherical. This was done in [Bi, O].

Definition 2.16. A *cospherical* Hopf algebra is a pair (H, t) , where H is a Hopf algebra and $t \in \text{Alg}(H, \mathbb{k})$ is such that

$$(2.20) \quad \mathcal{S}^2(x_{(1)})t(x_{(2)}) = t(x_{(1)})x_{(2)}, \quad x \in H,$$

$$(2.21) \quad \text{tr}_V((\text{id}_V \otimes t)\rho_V \vartheta) = \text{tr}_V((\text{id}_V \otimes t^{-1})\rho_V \vartheta), \quad \vartheta \in \text{End}_H(V),$$

⁵These control the passage from left to right integrals.

for all $V \in \text{Corep } H$. We say that $t \in \text{Alg}(H, \mathbb{k})$ is a *copivot* when it satisfies (2.20); pairs (H, t) with t a copivot are called *copivotal Hopf algebras*. A *cospherical element* is a copivot that fulfills (2.21). Let H be a cospherical Hopf algebra. Then the category $\text{Corep } H$ is spherical; in fact, the left and right traces are given by the sides of (2.21).

The set $\text{Alg}(H, \mathbb{k})$ is a subgroup of the group $\text{Hom}_*(H, \mathbb{k})$ of convolution-invertible linear functionals, which in turn acts on $\text{End}(H)$ on both sides. Hence (2.20) can be written as $\mathcal{S}^2 * t = t * \text{id}_H$ or else as $\mathcal{S}^2 = t * \text{id}_H * t^{-1}$. The copivot is not unique but it is determined up to multiplication by an element in $\text{Alg}(H, \mathbb{k})$ that centralizes id_H . The antipode of a copivotal Hopf algebra is bijective, with inverse given by $\mathcal{S}^{-1}(x) = \sum t^{-1}(x_{(1)})\mathcal{S}(x_{(2)})t(x_{(3)})$, $x \in H$.

The following statement is proved exactly as Proposition 2.1.

Proposition 2.17. *A copivotal Hopf algebra (H, t) is cospherical if and only if (2.21) holds for all simple H -comodules. \square*

Examples 2.18. (a) Assume that H is finite-dimensional. Then H is copivotal (resp., cospherical) iff H^* is pivotal (resp., spherical).

- (b) Any involutory Hopf algebra is cospherical with $t = \varepsilon$.
- (c) A copivotal Hopf algebra with involutive copivot is cospherical.
- (d) Condition (2.20) is multiplicative on x ; also, it holds for $x \in G(H)$.
- (e) Let H be a pointed Hopf algebra generated as an algebra by $G(H)$ and a family $(x_i)_{i \in I}$, where x_i is $(g_i, 1)$ skew-primitive. Assume that $g_i x_i g_i^{-1} = q_i x_i$, with $q_i \in \mathbb{k}^\times \setminus \{1\}$ for all $i \in I$. If $t \in \text{Alg}(H, \mathbb{k})$, then $t(x_i) = 0$, $i \in I$. Hence t is a copivot iff $t(g_i) = q_i^{-1}$, for all $i \in I$.
- (f) Let H be a pointed Hopf algebra as in item (e) and $t \in \text{Alg}(H, \mathbb{k})$ a copivot. Then H is cospherical iff $t(g) \in \{\pm 1\}$ for all $g \in G(H)$.
- (g) The notion of coribbon Hopf algebra is formally dual to the notion of ribbon Hopf algebra, see [H, LT]. Coribbon Hopf algebras are cospherical. For instance, the quantized function algebra $\mathcal{O}_q(G)$ of a semisimple algebraic group is cosemisimple and coribbon, when q is not a root of 1.

We recall now the construction of universal copivotal Hopf algebras.

Definition 2.19. [Bi] Let $F \in GL_n(k)$. The Hopf algebra $H(F)$ is the universal algebra with generators $(u_{ij})_{1 \leq i, j \leq n}$, $(v_{ij})_{1 \leq i, j \leq n}$ and relations

$$uv^t = v^t u = 1, \quad v F u^t F^{-1} = F u^t F^{-1} v = 1.$$

The comultiplication is determined by $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$, $\Delta(v_{ij}) = \sum_k v_{ik} \otimes v_{kj}$ and the antipode by $\mathcal{S}(u) = v^t$, $\mathcal{S}(v) = F u^t F^{-1}$. The Hopf algebra $H(F)$ is copivotal, the copivot being $t_F(u) = (F^{-1})^t$ and $t_F(v) = F$.

Let H be a Hopf algebra provided with $V \in \text{Corep } H$ of dimension n such that $V \cong V^{**}$. Then there exist a matrix $F \in GL_n(k)$, a coaction $\beta_V : V \rightarrow V \otimes H(F)$ and a Hopf algebra morphism $\pi : H(F) \rightarrow H$ such that $(\text{id}_V \otimes \pi)\beta_V = \rho_V$.

In conclusion, we would like to explore semisimple tensor categories arising from cospherical, not cosemisimple, Hopf algebras. The first step is the dual version of Theorem 2.7, which is proved exactly in the same way.

Theorem 2.20. *Let H be a cospherical Hopf algebra. Then the non-degenerate quotient $\underline{\text{Corep}} H$ of $\text{Corep } H$ is a completely reducible spherical tensor category, and $\text{Irr } \underline{\text{Corep}} H$ is in bijective correspondence with the set of isomorphism classes of indecomposable finite-dimensional H -comodules with non-zero quantum dimension.* \square

3. TILTING MODULES

The concept of tilting modules appeared in [BrB] and was extended to quasi-hereditary algebras in [Ri]. Observe that finite-dimensional quasi-hereditary Hopf algebras are semisimple. Indeed, quasi-hereditary algebras have finite global dimension, but a finite-dimensional Hopf algebra is a Frobenius algebra, hence it has global dimension 0 or infinite. Instead, the context where the recipe of tilting modules works is a suitable category of modules, or comodules, of an infinite dimensional Hopf algebra. The relevant examples are: algebraic semisimple groups over an algebraically closed field of positive characteristic (the representations are comodules over the Hopf algebra of rational functions), quantum groups at roots of one and the category \mathcal{O} over a semisimple Lie algebra [A1, AP, D1, GM, M]. The main features are:

- The suitable category of representations is not artinian, and the simple modules are parameterized by dominant weights; the set of dominant weights admits a total order that refines the usual partial order. To fit into the framework of quasi-hereditary algebras, subcategories of modules with weights in suitable subsets are considered; this allows to define Weyl modules $\Delta(\lambda)$, dual Weyl modules $\nabla(\lambda)$, and eventually tilting modules $T(\lambda)$, for λ a dominant weight. Usually these constructions are performed in an *ad-hoc* manner, not through quasi-hereditary algebras, albeit those corresponding to this situation are studied in the literature under the name of Schur algebras.
- The tensor product of two tilting modules and the dual of a tilting module are again tilting. The later statement is trivial, the former requires a delicate proof.
- There is an *alcove* inside the chamber defined by the positive roots and bounded by an affine hyperplane. If λ is in the alcove, then the simple module satisfies $L(\lambda) = \Delta(\lambda)$, hence it is the tilting $T(\lambda)$. The tilting modules $T(\lambda)$ outside the alcove are projective, hence have zero quantum dimension. Thus, the fusion category looked for is spanned by the tilting modules in the alcove.
- The fusion rules between the tilting modules is given by the celebrated Verlinde formula [V] or a modular version, see [AP, M].

We would like to adapt these arguments to categories of representations of certain Hopf algebras H arising from finite-dimensional Nichols of diagonal type. The Hopf algebra H would be the Drinfeld double, or a variation

thereof, of the bosonization of the corresponding Nichols algebra with a suitable abelian group. We would like to solve the following points:

- The set of irreducible objects in $\text{Rep } H$ (or some appropriate variation) should split as a filtered union $\text{Irr } H = \bigcup_{A \in \mathcal{A}} A$; each A spans an artinian subcategory where tilting modules \mathcal{T}_A can be computed.
- Define the category \mathcal{T}_H of tilting modules over H as the union of the various \mathcal{T}_A ; this should be a semisimple category.
- the category \mathcal{T}_H of tilting modules is stable by tensor products and duals.
- It is possible to determine which irreducible tilting modules have non-zero quantum dimension; there are a finite number of them.
- The fusion rules are expressed through a variation of the Verlinde formula.

Provided that these considerations are correct, the full subcategory of $\text{Rep } H$ generated by the indecomposable tilting modules with non-zero quantum dimension, is a fusion category. In this way, we hope to obtain new examples of non-integral fusion categories.

3.1. Quasi-hereditary algebras and tilting modules. Tilting modules work for our purpose because they span a completely reducible category already in $\text{Rep } H$. We think it is worthwhile to recall the main definitions of the theory of (partial) tilting modules over quasi-hereditary algebras, due to Ringel [Ri]. A full exposition is available in [D2].

Let A be an artin algebra. Consider a family $\Theta = (\Theta(1), \dots, \Theta(n))$ of A -modules such that

$$(3.1) \quad \text{Ext}_A^1(\Theta(j), \Theta(i)) = 0, \quad j \geq i.$$

We denote by $\mathcal{F}(\Theta)$ the full subcategory of $A\text{-Mod}$ with objects M that admit a filtration with sub-factors in Θ . We fix a numbering (that is, a total order) of $\text{Irr } A$: $L(1), \dots, L(n)$. We set

$$\begin{aligned} P(i) &= \text{projective cover of } L(i), \\ Q(i) &= \text{injective hull of } L(i), \\ \Delta(i) &= P(i)/U(i), \quad \text{where } U(i) = \sum_{j>i} \sum_{\alpha \in \text{Hom}(P(j), P(i))} \text{Im } \alpha, \\ \nabla(i) &= \bigcap_{j>i} \bigcap_{\beta \in \text{Hom}(Q(i), Q(j))} \ker \beta, \end{aligned}$$

$1 \leq i \leq n$. Let $\Delta = (\Delta(1), \dots, \Delta(n))$, $\nabla = (\nabla(1), \dots, \nabla(n))$; these satisfy (3.1) and then [Ri, Theorem 1] applies to them.

Definition 3.1. The artin algebra A is *quasi-hereditary* provided that ${}_A A \in \mathcal{F}(\Delta)$, and $L(i)$ has multiplicity one in $\Delta(i)$, $1 \leq i \leq n$.

Remark 3.2. Quasi-hereditary algebras were introduced by Cline, Parshall and Scott, see e. g. [CPS]. There are some alternative definitions.

- (a) An ideal J of an artin algebra A is *hereditary* provided that
- $J \in A\text{-Mod}$ is projective,
 - $\text{Hom}_A(J, A/J) = 0$ (Ringel assumes $J^2 = J$ instead of this),
 - $JNJ = 0$, where N is the radical of A .
- It can be shown that A is quasi-hereditary iff there exists a chain of ideals $A = J_0 > J_1 > \cdots > J_m = 0$ with J_i/J_{i+1} hereditary in A/J_{i+1} .
- (b) Also, A is quasi-hereditary iff $A\text{-Mod}$ is a highest weight category, that is the following holds for all i :
- $Q(i)/\nabla(i) \in \mathcal{F}(\nabla)$.
 - If $(Q(i)/\nabla(i) : \nabla(j)) \neq 0$, then $j > i$.

For completeness, we include the definitions of tilting, cotilting and basic modules, see e.g. [Ri] and its bibliography. First, a module T is tilting provided that

- it has finite projective dimension;
- $\text{Ext}^i(T, T) = 0$ for all $i \geq 1$;
- for any projective module P , there should exist an exact sequence $0 \rightarrow P \rightarrow T_0 \rightarrow \cdots \rightarrow T_m \rightarrow 0$, with all T_j in the additive subcategory generated by T , denoted $\text{add } T$.

Second, a cotilting module should have

- finite injective dimension;
- $\text{Ext}^i(T, T) = 0$ for all $i \geq 1$;
- for any injective module I , there should exist an exact sequence $0 \rightarrow T_m \rightarrow \cdots \rightarrow T_0 \rightarrow I \rightarrow 0$, with all T_j in $\text{add } T$.

Lastly, a basic module is one with no direct summands of the form $N \oplus N$, with $N \neq 0$.

Assume that A is a quasi-hereditary algebra and consider the full subcategory $\mathcal{T} = \mathcal{T}_A = \mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$ of (partial) tilting modules. It was shown in [Ri, Theorem 5] that— for a quasi-hereditary algebra— there is a unique basic module T , that is tilting and cotilting (in the sense just above), and such that \mathcal{T} coincides with $\text{add } T$. The relation between this T and the partial tilting modules is clarified by the following result.

Theorem 3.3. [Ri, Corollary 2, Proposition 5] *There exist indecomposable (partial) tilting modules $T(i) \in \mathcal{T}$, $1 \leq i \leq n$, with the following properties:*

- Any indecomposable tilting module is isomorphic to one of them.
- $T = T(1) \oplus \cdots \oplus T(n)$ is the tilting module mentioned above.
- There are exact sequences

$$\begin{aligned} 0 \longrightarrow \Delta(i) \xrightarrow{\beta(i)} T(i) \longrightarrow X(i) \longrightarrow 0, \\ 0 \longrightarrow Y(i) \longrightarrow T(i) \xrightarrow{\gamma(i)} \nabla(i) \longrightarrow 0, \end{aligned}$$

where $X(i) \in \mathcal{F}(\{\Delta(j) : j < i\})$, $Y(i) \in \mathcal{F}(\{\nabla(j) : j < i\})$, $\beta(i)$ is a left $\mathcal{F}(\nabla)$ -approximation and $\gamma(i)$ is a right $\mathcal{F}(\Delta)$ -approximation, $1 \leq i \leq n$ (see [Ri] for undefined notions). \square

3.2. Induced and produced. Let $B \hookrightarrow A$ be an inclusion of algebras. We denote by Res_B^A the restriction functor from the category $A\text{-Mod}$ to $B\text{-Mod}$.

3.2.1. The induced and produced modules of $T \in B\text{-Mod}$ are

$$(3.2) \quad \text{Ind}_B^A T = A \otimes_B T, \quad \text{Pro}_B^A T = \text{Hom}_B(A, T).$$

These are equipped with morphisms of B -modules $\iota : T \hookrightarrow \text{Ind}_B^A T$, given by $\iota(t) = 1 \otimes t$ for $t \in T$, and $\pi : \text{Pro}_B^A T \twoheadrightarrow T$, given by $\pi(f) = f(1)$ for $f \in \text{Hom}_B(A, T)$. The following properties are well-known.

- (a) $\text{Hom}_B(T, \text{Res}_B^A M) \simeq \text{Hom}_A(\text{Ind}_B^A T, M)$; that is, induction is left adjoint to restriction.
- (b) For every $S \in \text{Irr } A$ there exists $T \in \text{Irr } B$ such that S is a quotient of $\text{Ind}_B^A T$.
- (c) $\text{Hom}_B(\text{Res}_B^A N, T) \simeq \text{Hom}_A(N, \text{Pro}_B^A T)$; that is, production (also called coinduction) is right adjoint to restriction.
- (d) For every $S \in \text{Irr } A$ there exists $T \in \text{Irr } B$ such that S is a submodule of $\text{Pro}_B^A T$.

3.2.2. Assume that A is a finite B -module. Then Res_B^A , Ind_B^A and Pro_B^A restrict to functors (denoted by the same name) between the categories $A\text{-mod}$ and $B\text{-mod}$ of finite-dimensional modules; *mutatis mutandis*, the preceding points hold in this context. Assume also that there exists a *contravariant* \mathbb{k} -linear functor $\mathcal{D} : A\text{-mod} \rightarrow A\text{-mod}$ such that $\mathcal{D}(B\text{-mod}) \subseteq B\text{-mod}$ and admits a quasi-inverse $\mathcal{E} : A\text{-mod} \rightarrow A\text{-mod}$, so that \mathcal{D} is an equivalence of categories. It follows at once from the universal properties that

$$(3.3) \quad \text{Ind}_B^A T \simeq \mathcal{D}(\text{Pro}_B^A \mathcal{E}T) \simeq \mathcal{E}(\text{Pro}_B^A \mathcal{D}T),$$

$$(3.4) \quad \text{Pro}_B^A T \simeq \mathcal{D}(\text{Ind}_B^A \mathcal{E}T) \simeq \mathcal{E}(\text{Ind}_B^A \mathcal{D}T).$$

Hence $\mathcal{D}(\text{Ind}_B^A T) \simeq \text{Pro}_B^A \mathcal{D}T$, $\mathcal{D}(\text{Pro}_B^A T) \simeq \text{Ind}_B^A \mathcal{D}T$, and so on.

In this setting, consider the following conditions:

$$(3.5) \quad \text{For every } S \in \text{Irr } A, \exists \text{ a unique } T \in \text{Irr } B \text{ such that } \text{Ind}_B^A T \twoheadrightarrow S.$$

$$(3.6) \quad \text{For every } S \in \text{Irr } A, \exists \text{ a unique } U \in \text{Irr } B \text{ such that } S \hookrightarrow \text{Pro}_B^A U.$$

$$(3.7) \quad \text{The head of } \text{Ind}_B^A T \text{ is simple for every } T \in \text{Irr } B.$$

$$(3.8) \quad \text{The socle of } \text{Pro}_B^A U \text{ is simple for every } U \in \text{Irr } B.$$

Then (3.5) \iff (3.6) and (3.7) \iff (3.8). If all these conditions hold, then for any $T \in \text{Irr } B$, there exists a unique $U \in \text{Irr } B$ such that

$$\text{Ind}_B^A T \twoheadrightarrow S \hookrightarrow \text{Pro}_B^A U,$$

where S is the head of $\text{Ind}_B^A T$ and the socle of $\text{Pro}_B^A U$. We set $U = w_0(T)$.

3.2.3. Let K be a Hopf subalgebra of a Hopf algebra H , with H finite over K . Then

$$\mathrm{Ind}_K^H T \simeq (\mathrm{Pro}_K^H * T)^* \simeq *(\mathrm{Pro}_K^H T^*), \quad \mathrm{Pro}_K^H T \simeq (\mathrm{Ind}_K^H * T)^* \simeq *(\mathrm{Ind}_K^H T^*).$$

If H is pivotal, then these formulae are simpler because the left and right duals coincide.

3.2.4. Let \mathfrak{g} be a simple Lie algebra, \mathfrak{b} a Borel subalgebra and q a root of unity of odd order, relatively prime to 3 when \mathfrak{g} is of type G_2 . Let $H = \mathcal{U}_q(\mathfrak{g})$ be the Lusztig's q -divided power quantized enveloping algebra and $K = \mathcal{U}_q(\mathfrak{b})$. Let \mathcal{C} be the category of finite-dimensional H -modules of type 1, see [APW], and $\mathcal{C}_\mathfrak{b}$ the analogous category of K -modules. Then there are induced and produced functors $\mathrm{Ind}_K^H, \mathrm{Pro}_K^H : \mathcal{C}_\mathfrak{b} \rightarrow \mathcal{C}$. Then the Weyl and dual Weyl modules are the produced and induced modules of the simple objects in $\mathcal{C}_\mathfrak{b}$, parameterized conveniently by highest weights. This allows to define Weyl and dual Weyl filtrations and tilting modules; instead of appealing to Theorem 3.3, one establishes the semisimplicity of the category of tilting modules by establishing crucial cohomological results, see [AP] for details.

3.3. Finite-dimensional Nichols algebras of diagonal type. We continue the analysis started in Subsection 1.4.

3.3.1. Let $\theta \in \mathbb{N}$ and $\mathbb{I} = \{1, \dots, \theta\}$. Let Λ be a free abelian group with basis $\alpha_1, \dots, \alpha_\theta$. Let \leq be the partial order in Λ defined by

$$(3.9) \quad \lambda \leq \mu \iff \mu - \lambda \in \Lambda^+ := \sum_{i \in \mathbb{I}} \mathbb{N}_0 \alpha_i.$$

Let us fix a \mathbb{Z} -linear injective map $E : \Lambda \rightarrow \mathbb{R}$ such that $E(\alpha_i) > 0$ for all $i \in \mathbb{I}$. This induces a total order on Λ by $\lambda \preceq \mu \iff E(\lambda) \leq E(\mu)$; clearly, $\lambda \leq \mu$ implies $\lambda \preceq \mu$.

Given a Λ -graded vector space $M = \bigoplus_{\lambda \in \Lambda} M_\lambda$, the λ 's such that $M_\lambda \neq 0$ are called the weights of M ; the set of all its weights is denoted $\Pi(M)$.

3.3.2. Let $(q_{ij})_{i,j \in \mathbb{I}}$ be a symmetric matrix with entries in \mathbb{k}^\times . Let (V, c) be a braided vector space of diagonal type with matrix $(q_{ij})_{i,j \in \mathbb{I}}$, with respect to a basis $(v_i)_{i \in \mathbb{I}}$. The Nichols algebra $\mathcal{B}(V)$ has a Λ -grading determined by $\deg v_i = \alpha_i$, $i \in \mathbb{I}$. By [Kh], there exists an ordered set \bar{S} of homogeneous elements of $T(V)$ and a function $h : \bar{S} \rightarrow \mathbb{N} \cup \{\infty\}$ such that:

- The elements of \bar{S} are hyperletters in $(v_i)_{i \in \mathbb{I}}$.
- The projection $T(V) \rightarrow \mathcal{B}(V)$ induces a bijection of \bar{S} with its image S . Denote also by $h : S \rightarrow \mathbb{N} \cup \{\infty\}$ the induced function.
- The following elements form a basis of $\mathcal{B}(V)$:

$$(3.10) \quad \{s_1^{e_1} \dots s_t^{e_t} : t \in \mathbb{N}_0, \quad s_1 > \dots > s_t, \quad s_i \in S, \quad 0 < e_i < h(s_i)\}.$$

When S is finite, two distinct elements in S have different degree, and we can label the elements in S with a finite subset Δ_+^V of Λ_+ ; this is instrumental to define the root system \mathcal{R} of (V, c) [HS].

Let W be another braided vector space of diagonal type with matrix $(q_{ij}^{-1})_{i,j \in \mathbb{I}}$, with respect to a basis w_1, \dots, w_θ ; we shall consider the Λ -grading on the Nichols algebra $\mathcal{B}(W)$ determined by $\deg w_i = -\alpha_i$, $i \in \mathbb{I}$.

We assume from now on that $\dim \mathcal{B}(V) < \infty$; hence $\dim \mathcal{B}(W) < \infty$. Under this assumption, the connected components of (q_{ij}) belong to the list given in [He1]. An easy consequence is that $q_{ii} \neq 1$ and q_{ij} is a root of 1 for all $i, j \in \mathbb{I}$; this last claim follows because the matrix (q_{ij}) is assumed to be symmetric. Also, S is finite and h takes values in \mathbb{N} . Thus (3.10) says that

$$\Pi(\mathcal{B}(V)) = \left\{ \sum_{s \in S} e_s \deg s, \quad 0 \leq e_s < h(s) \right\}.$$

Note that $0 \leq \alpha \leq \varrho$ for all $\alpha \in \Pi(\mathcal{B}(V))$, where

$$(3.11) \quad \varrho = \sum_{s \in S} (h(s) - 1) \deg s = \deg \mathcal{B}^{\text{top}}(V) \in \Pi(\mathcal{B}(V)).$$

3.3.3. A pre-Nichols algebra of V is any graded braided Hopf algebra \mathfrak{T} intermediate between $\mathcal{B}(V)$ and $T(V)$: $T(V) \twoheadrightarrow \mathfrak{T} \twoheadrightarrow \mathcal{B}(V)$ (Masuoka). The defining relations of the Nichols algebra $\mathcal{B}(V) = T(V)/\mathcal{J}(V)$ are listed as (40), \dots , (68) in [Ang3, Theorem 3.1]. Now we observe that, since $\dim \mathcal{B}(V) < \infty$, the following integers exist:

$$(3.12) \quad -a_{ij} := \min \{n \in \mathbb{N}_0 : (n+1)_{q_{ii}}(1 - q_{ii}^n q_{ij}^2) = 0\}$$

for all $j \in \mathbb{I} - \{i\}$. Set also $a_{ii} = 2$. The *distinguished pre-Nichols algebra* of V is $\widehat{\mathcal{B}}(V) = T(V)/\widehat{\mathcal{J}}(V) = \bigoplus_{n \in \mathbb{N}_0} \widehat{\mathcal{B}}^n(V)$, where $\widehat{\mathcal{J}}(V)$ is the ideal of $T(V)$ generated by

- relations (41), \dots , (68) in [Ang3, Theorem 3.1],
- the quantum Serre relations $(\text{ad}_c x_i)^{1-a_{ij}} x_j$ for those vertices such that $q_{ii}^{a_{ij}} = q_{ij} q_{ji}$.

The ideal $\widehat{\mathcal{J}}(V)$ was introduced in [Ang3], see the paragraph after Theorem 3.1; $\widehat{\mathcal{J}}(V)$ is a braided bi-ideal of $T(V)$, so that there is a projection of braided Hopf algebras $\widehat{\mathcal{B}}(V) \twoheadrightarrow \mathcal{B}(V)$ [Ang3, Proposition 3.3].

Definition 3.4. We say that $p \in \{1, \dots, \theta\}$ is a *Cartan vertex* if, for every $j \neq p$, $q_{pp}^{a_{pj}} = q_{pj} q_{jp}$. In such case, $\text{ord } q_{pp} \geq 1 - a_{pj}$ by hypothesis.

Clearly the projection $T(V) \rightarrow \widehat{\mathcal{B}}(V)$ induces a bijection of \bar{S} with its image \widehat{S} . Denote again by h the induced function. Let $\widehat{h} : \widehat{S} \rightarrow \mathbb{N} \cup \{\infty\}$ be the function given by

$$\widehat{h}(s) = \begin{cases} \infty, & \text{if } s \text{ is conjugated to a Cartan vertex} \\ h(s), & \text{otherwise.} \end{cases}$$

Then the following set is a basis of $\widehat{\mathcal{B}}(V)$, see the end of the proof of [Ang3, Theorem 3.1]:

$$(3.13) \quad \{s_1^{e_1} \dots s_t^{e_t} : t \in \mathbb{N}_0, \quad s_1 > \dots > s_t, \quad s_i \in \widehat{S}, \quad 0 < e_i < \widehat{h}(s_i)\}.$$

3.3.4. The *Lusztig algebra* $\mathcal{L}(V)$ of V is the graded dual of $\widehat{\mathcal{B}}(V)$, that is $\mathcal{L}(V) = \bigoplus_{n \in \mathbb{N}_0} \mathcal{L}^n(V)$, where $\mathcal{L}^n(V) = \widehat{\mathcal{B}}^n(V)^*$. The Lusztig algebra $\mathcal{L}(V)$ of V is the analogue of the q -divided powers algebra introduced in [L1, L2].

3.4. The small quantum groups. We consider finite-dimensional pointed Hopf algebras attached to the matrix $(q_{ij})_{i,j \in \mathbb{I}}$, analogues of the small quantum groups or Frobenius-Lusztig kernels. We need the following additional data: A finite abelian group Γ , provided with elements $g_1, \dots, g_\theta \in \Gamma$ and characters $\chi_1, \dots, \chi_\theta \in \text{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{k}^\times)$ such that

$$(3.14) \quad \chi_j(g_i) = q_{ij}, \quad i, j \in \mathbb{I}.$$

We define a structure of Yetter-Drinfeld module over $\mathbb{k}\Gamma$ on $W \oplus V$ by

$$(3.15) \quad v_i \in V_{g_i}^{\chi_i}, \quad w_i \in W_{g_i}^{\chi_i^{-1}}, \quad i \in \mathbb{I}.$$

Let \mathfrak{u} be the Hopf algebra $T(W \oplus V) \# \mathbb{k}\Gamma / \mathfrak{I}$, where \mathfrak{I} is the ideal generated by $\mathcal{J}(V)$, $\mathcal{J}(W)$ and the relations

$$(3.16) \quad v_i w_j - \chi_j^{-1}(g_i) w_j v_i - \delta_{ij}(g_i^2 - 1) \quad i, j \in \mathbb{I}.$$

This is a pointed quasi-triangular Hopf algebra with $\dim \mathfrak{u} = |\Gamma| \dim \mathcal{B}(V)^2$. The freedom to choose the abelian group Γ allows more flexibility, but otherwise this is very close to the small quantum groups (with more general Nichols algebras). By choosing Γ appropriately, \mathfrak{u} is a spherical Hopf algebra. Let \mathfrak{u}^b (resp. \mathfrak{u}^-) be the subalgebra of \mathfrak{u} generated by v_1, \dots, v_θ and Γ (resp. w_1, \dots, w_θ). Consider the morphisms of algebras $\rho_V : \mathcal{B}(V) \rightarrow \mathfrak{u}$, $\rho_W : \mathcal{B}(W) \rightarrow \mathfrak{u}$ and $\rho_\Gamma : \mathbb{k}\Gamma \rightarrow \mathfrak{u}$, given by $\rho_V(v_i) = v_i$, $\rho_W(w_i) = w_i$, $\rho_\Gamma(g_i) = g_i$, $i \in \mathbb{I}$. Then

- (a) $\rho_V, \rho_W, \rho_\Gamma$ give rise to isomorphisms $\mathcal{B}(W) \simeq \mathfrak{u}^-$, $\mathcal{B}(V) \# \mathbb{k}\Gamma \simeq \mathfrak{u}^b$.
- (b) The map $\mathcal{B}(V) \otimes \mathcal{B}(W) \otimes \mathbb{k}\Gamma \rightarrow \mathfrak{u}$, $v \otimes w \otimes g \mapsto \rho_V(v) \rho_W(w) \rho_\Gamma(g)$ is a coalgebra isomorphism.
- (c) The multiplication maps $\mathfrak{u}^- \otimes \mathfrak{u}^b \rightarrow \mathfrak{u}$, $\mathfrak{u}^b \otimes \mathfrak{u}^- \rightarrow \mathfrak{u}$ are linear isomorphisms.

See [Ma, Theorem 5.2], [ARS, Corollary 3.8]. Now suppose that one would like to define tilting modules over \mathfrak{u} , ignoring that this is not a quasi-hereditary algebra. Inducing from \mathfrak{u}^b , we see that simple modules correspond to characters of Γ ; but the set of simple modules could not be totally ordered and (3.6) and (3.7) do not necessarily hold. A first approach to remedy this that might come to the mind is to assume the following extra hypothesis: *There exists a \mathbb{Z} -bilinear form $\langle \cdot, \cdot \rangle : \Gamma \times \Lambda \rightarrow \mathbb{k}^\times$ such that*

$$(3.17) \quad \langle g_i, \alpha_j \rangle = q_{ij}, \quad i, j \in \mathbb{I}.$$

Then we may consider a category that is an analogue of category of representations of the algebraic group $G_1 T$ in positive characteristic, or else of its

quantum analogue in the literature of quantum groups. Let $\mathcal{C}_{\mathfrak{u}}$ be the category of finite-dimensional \mathfrak{u} -modules M with a Λ -grading $M = \bigoplus_{\lambda \in \Lambda} M_{\lambda}$, compatible with the action of \mathfrak{u} in the sense

$$(3.18) \quad M_{\lambda} = \{m \in M : g \cdot m = \langle g, \lambda \rangle m, \quad g \in \Gamma\}, \quad \lambda \in \Lambda,$$

$$(3.19) \quad v_i \cdot M_{\lambda} = M_{\lambda + \alpha_i}, \quad w_i \cdot M_{\lambda} = M_{\lambda - \alpha_i}, \quad \lambda \in \Lambda, \quad i \in \mathbb{I}.$$

Morphisms in $\mathcal{C}_{\mathfrak{u}}$ preserve both the action of \mathfrak{u} and the grading by Λ . The category $\mathcal{C}_{\mathfrak{u}^b}$ is defined analogously. Both categories $\mathcal{C}_{\mathfrak{u}}$ and $\mathcal{C}_{\mathfrak{u}^b}$ are spherical tensor categories (up to an appropriate choice of Γ), with duals defined in the obvious way. There are functors $\text{Res}_{\mathfrak{u}^b}^{\mathfrak{u}}$, $\text{Ind}_{\mathfrak{u}^b}^{\mathfrak{u}}$ and $\text{Pro}_{\mathfrak{u}^b}^{\mathfrak{u}}$ between the categories $\mathcal{C}_{\mathfrak{u}}$ and $\mathcal{C}_{\mathfrak{u}^b}$; indeed

$$\text{Ind}_{\mathfrak{u}^b}^{\mathfrak{u}} T = \mathfrak{u} \otimes_{\mathfrak{u}^b} T \simeq \mathcal{B}(W) \otimes T, \quad \text{Pro}_{\mathfrak{u}^b}^{\mathfrak{u}} T = \text{Hom}_{\mathfrak{u}^b}(\mathfrak{u}, T) \simeq \text{Hom}(\mathcal{B}(W), T),$$

so that the grading in $\text{Ind}_{\mathfrak{u}^b}^{\mathfrak{u}} T$, resp. $\text{Pro}_{\mathfrak{u}^b}^{\mathfrak{u}} T$, arises from that of $\mathcal{B}(W) \otimes T$, resp. $\text{Hom}(\mathcal{B}(W), T)$.

Given $\lambda \in \Lambda$, we denote by \mathbb{k}_{λ} the vector space with generator $\mathbf{1}_{\lambda}$, considered as object in $\mathcal{C}_{\mathfrak{u}^b}$ by

$$\deg \mathbf{1}_{\lambda} = \lambda, \quad g \cdot \mathbf{1}_{\lambda} = \langle g, \lambda \rangle \mathbf{1}_{\lambda}, \quad g \in \Gamma, \quad v \cdot \mathbf{1}_{\lambda} = 0, \quad v \in V.$$

Note that $\mathbb{k}_{\lambda} \simeq \mathbb{k}_{\mu}$ in \mathfrak{u}^b -mod whenever $\lambda - \mu \in \Gamma^{\perp}$, but they are not isomorphic as objects in $\mathcal{C}_{\mathfrak{u}^b}$ unless $\lambda = \mu$. Clearly, $\text{Irr } \mathcal{C}_{\mathfrak{u}^b} = \{\mathbb{k}_{\lambda} : \lambda \in \Lambda\}$.

Consider the modules $\text{Pro}_{\mathfrak{u}^b}^{\mathfrak{u}}(\mathbb{k}_{\lambda})$ and $\Delta(\lambda) := \text{Ind}_{\mathfrak{u}^b}^{\mathfrak{u}}(\mathbb{k}_{\lambda})$. We know

$$(3.20) \quad \Pi(\Delta(\lambda)) = \{\lambda - \alpha : \alpha \in \Pi(\mathcal{B}(V))\},$$

$$(3.21) \quad \Pi(\text{Pro}_{\mathfrak{u}^b}^{\mathfrak{u}}(\mathbb{k}_{\lambda})) = \{\lambda + \alpha : \alpha \in \Pi(\mathcal{B}(V))\}.$$

Thus, $\Delta(\lambda)$ has a highest weight λ and a lowest weight $\lambda - \varrho$, both of multiplicity 1; and $\text{Pro}_{\mathfrak{u}^b}^{\mathfrak{u}}(\mathbb{k}_{\lambda})$ has a highest weight $\lambda + \varrho$ and a lowest weight λ , both of multiplicity 1. For convenience, set $\nabla(\lambda) = \text{Pro}_{\mathfrak{u}^b}^{\mathfrak{u}}(\mathbb{k}_{\lambda - \varrho}) \simeq \Delta(-\lambda + \varrho)^*$, since $\mathbb{k}_{\lambda}^* \simeq \mathbb{k}_{-\lambda}$.

The statements (a), (b), (c) and (d) in 3.2.1 above carry over to the present setting. We claim that (3.5), (3.6), (3.7) and (3.8) also hold here.

Indeed, let $S \in \text{Irr } \mathcal{C}_{\mathfrak{u}}$ and $\lambda, \mu \in \Pi(S)$ such that $\mu \preceq \tau \preceq \lambda$ for all $\tau \in \Pi(S)$. If $m \in S_{\lambda} - 0$, then $v_i \cdot m = 0$ for all $i \in \mathbb{I}$ by (3.19), hence $\mathbb{k}m \simeq \mathbb{k}_{\lambda}$ and we have $\Delta(\lambda) \twoheadrightarrow S$ and $\Pi(S) \subseteq \{\lambda - \alpha : \alpha \in \Pi(\mathcal{B}(V))\}$ by (3.20). Moreover, if $\Delta(\lambda') \twoheadrightarrow S$, then $\lambda \in \Pi(S) \subseteq \{\lambda' - \alpha : \alpha \in \Pi(\mathcal{B}(V))\}$, hence $\lambda' = \lambda$, showing (3.5). The proof of (3.7) is standard: $\Delta(\lambda)$ has a unique maximal submodule, which is the sum of all submodules intersecting trivially $\Delta(\lambda)_{\lambda}$. Now (3.6) and (3.8) follow by duality, so that $S \hookrightarrow \nabla(\mu)$.

In conclusion we have the following standard result.

Proposition 3.5. *If $E(\lambda) := \text{head of } \Delta(\lambda)$, then $\text{Irr } \mathcal{C}_{\mathfrak{u}} = \{E(\lambda) : \lambda \in \Lambda\}$. If $\mu \in \Gamma^{\perp}$, then $\dim E(\mu) = 1$ and $E(\mu) \otimes E(\lambda) \simeq E(\lambda) \otimes E(\mu) \simeq E(\lambda + \mu)$. There is a bijection $w_0 : \Lambda \rightarrow \Lambda$ such that $E(\lambda) := \text{socle of } \nabla(w_0(\lambda))$. \square*

The modules $\Delta(\lambda)$, resp. $\nabla(\lambda)$, are called the Weyl modules, resp. the dual Weyl modules. We may then go on and define good and Weyl filtrations, and tilting modules. However, it is likely that tilting modules are projective, thus with 0 quantum dimension, as is the case for G_1T , see [A2, 3.4], [Jn].

3.5. Generalized quantum groups. The next idea is to replace Γ by an infinite abelian group Q , perhaps free of finite rank, and the Nichols algebras $\mathcal{B}(V)$, $\mathcal{B}(W)$ by the distinguished pre-Nichols algebras $\widehat{\mathcal{B}}(V)$, $\widehat{\mathcal{B}}(W)$. Namely, we assume that Q is provided with elements K_1, \dots, K_θ and characters $\Upsilon_1, \dots, \Upsilon_\theta \in \text{Hom}_{\mathbb{Z}}(Q, \mathbb{k}^\times)$ such that $\Upsilon_j(K_i) = q_{ij}$, $i, j \in \mathbb{I}$. Then $W \oplus V$ is also a Yetter-Drinfeld module over $\mathbb{k}Q$ by $v_i \in V_{K_i}^{\Upsilon_i}$, $w_i \in W_{K_i}^{\Upsilon_i^{-1}}$, $i \in \mathbb{I}$. Let $U(V) = T(W \oplus V) \# \mathbb{k}Q / \widehat{\mathcal{J}}$ where $\widehat{\mathcal{J}}$ is the ideal generated by $\widehat{\mathcal{J}}(V)$, $\widehat{\mathcal{J}}(W)$ and the relations

$$(3.22) \quad v_i w_j - \chi_j^{-1}(g_i) w_j v_i - \delta_{ij}(g_i^2 - 1) \quad i, j \in \mathbb{I}.$$

This Hopf algebra, for a suitable Q , was introduced in [Ang3]; it is the analogue of the quantized enveloping algebra at a root of one for $(q_{ij})_{i,j \in \mathbb{I}}$ in the version of [dCP]. It also has a triangular decomposition similar as in the case of \mathfrak{u} . Furthermore, there are so-called Lusztig isomorphisms, because they generalize the braid group representations defined by Lusztig, see e. g. [L3]. Actually, the definition of the ideal $\widehat{\mathcal{J}}$ in [Ang3] was designed to have (a) a braided bi-ideal, and (b) the Lusztig automorphisms at the level of $U(V)$, generalizing results from [He2]. More precisely, the situation is as follows.

We assume that Q and Γ are accurately chosen and that there is a group epimorphism $Q \rightarrow \Gamma$. Given $i \in \mathbb{I}$, we define the i -th reflection of (V, c) . Define $s_i \in \text{Aut } \Lambda$ by $s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i$, see (3.12). Then $s_i(V, c) = (V, c')$, where C' is the braiding of diagonal type with matrix $(\tilde{q}_{rs})_{r,s \in \mathbb{I}}$. Here $\tilde{q}_{rs} = (s_i(\alpha_r) | s_i(\alpha_s))$; we omit the mention to the braidings c, c' , etc. Then

- There are algebra isomorphisms $T_i, T_i^- : \mathfrak{u}(V) \rightarrow \mathfrak{u}(s_i V)$, such that $T_i T_i^- = T_i^- T_i = \text{id}_{\mathfrak{u}(V)}$ [He2, Theorem 6.12].
- There are algebra isomorphisms $T_i, T_i^- : U(V) \rightarrow U(s_i V)$, such that $T_i T_i^- = T_i^- T_i = \text{id}_{U(V)}$ [Ang3, Proposition 3.26], compatible with those of $\mathfrak{u}(V)$.

There is a Λ -grading on $T(W \oplus V) \# \mathbb{k}Q$ given by $\deg \gamma = 0$, $\gamma \in Q$, $\deg v_i = \alpha_i = -\deg w_i$, $i \in \mathbb{I}$; it extends to gradings of $\mathfrak{u}(V)$ and $U(V)$. Hence we may consider categories \mathcal{C}_U and so on. However, the Hopf algebra $U(V)$ has a large center Z and is actually finite over it. Thus, it seems that its representation theory should be addressed with the methods of [dCP, dCPRR].

It remains a third tentative: to repeat the above considerations replacing the distinguished pre-Nichols algebras $\widehat{\mathcal{B}}(V)$, $\widehat{\mathcal{B}}(W)$ by the Lusztig algebras $\mathcal{L}(V)$, $\mathcal{L}(W)$. We hope to address this in future publications.

Acknowledgements. We thank Olivier Mathieu for a course on tilting modules given at the University of Córdoba in November 2009; and also Alain Bruguières, Flavio Coelho, Juan Cuadra, Michael Müger, Dmitri Nikschy, Claus Ringel and Yorck Sommerhäuser for very useful conversations.

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N. A., I. A., A. G. I., C. V.: FAMAF-CIEM (CONICET), UNIVERSIDAD NACIONAL DE CÓRDOBA, MEDINA ALLENDE S/N, CIUDAD UNIVERSITARIA, 5000 CÓRDOBA, REPÚBLICA ARGENTINA.

E-mail address: (andrus|angiono|aigarcia|vay)@famaf.unc.edu.ar

B. T.: UNIVERSIDAD DE ALMERÍA, DPTO. ÁLGEBRA Y ANÁLISIS MATEMÁTICO. E04120 ALMERÍA, SPAIN

E-mail address: btorrecci@ual.es